NASA TT F-10,494

N67 13834
(ACCESSION NUMBER)
9
(PAGES)
(NASA CR OR TMX OR AD NUMBER)
(THRU)
1
(CODE)
12
(CATEGORY)
FACILITY FORM 602

THE CALCULATION OF FLOW IN A PLANE LAVAL NOZZLE

Ya. I. Alikhashkin, A. P. Favorskiy, P. I. Chushkin

Translation of "Oraschete techeniya v ploskom sople lavalya". Zhurnal Vychisletel'noy Matematiki i Matematicheskoy Fiziki, Vol. 3, pp. 1130-1134, 1963.

GPO PRICE $
CFSTI PRICE(S) $
Hard copy (HC) 1.00
Microfiche (MF) .50
ff 653 July 65

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON DECEMBER 1966

THE CALCULATION OF FLOW IN A PLANE LAVAL NOZZLE

Ya. I. Alikhashkin, A. P. Favorskiy, P. I. Chushkin

A method of calculation is presented for a potential uniform flow of a perfect gas in the subsonic and transonic part of a plane symmetrical Laval nozzle with a given form of contour.

Let us examine the calculation of potential uniform flow of a perfect gas in the subsonic and transonic part of a plane symmetrical Laval nozzle with a given form of contour. We will assume that the subsonic portion of the nozzle extends to infinity and that the walls of the nozzle are either parallel or diverge at a certain angle. /1130*

Let us study such a case where, under a certain expansion of gas through the nozzle along the entire narrowest (critical) cross-section of the nozzle, sonic velocities are attained; corresponding values of divergence and the form of the sonic line are determined in the result of the calculation. For the solution of this problem we will apply the numerical method of integral relations [1], with the help of which a whole series of two-dimensional problems in gas dynamics was effectively solved [2]. The solution of the current problem under the first approximation of the method of integral relations was found earlier (see [2]). The solution in the general case is given below and examples are cited of calculations under the first and second approximations.

Let us introduce the origin of a cartesian coordinate system x,y in the critical section of the nozzle on its axis, while we orient the x axis in the direction of flow. In view of the symmetry of the nozzle, it is sufficient to investigate only its upper half ($y \geqslant 0$).

As the basic equations of the problem, let us take the equations of Chaplygin

$$\frac{\partial F}{\partial \varphi}+\frac{\partial \vartheta}{\partial \psi}=0, \quad \frac{\partial \vartheta}{\partial \varphi}-\frac{\partial f}{\partial \psi}=0. \tag{1}$$

The velocity potential φ and the stream function ψ act here as the independent variables, while the dependent variables are the velocity modulus V and ϑ its angle of inclination to the axis of the nozzle. Let us make use of dimensionless units, relating the velocity to the maximum adiabatic velocity and the density to the stagnation density. Then, it is obvious that $a_*^2 = (\varkappa - 1)/(\varkappa + 1)$, /1131

*/Numbers in the margin indicate pagination of the original foreign text.

where a* is the critical velocity of sound and $\varkappa$ is the index of the adiabatic curve. In equations (1), F and f are defined by the velocity functions

$$F = \int \frac{a_*^2 - V^2}{V a_*^2 \rho^{\varkappa}} dV, \qquad f = \int \frac{\rho\, dV}{V}, \qquad \rho = (1 - V^2)^{1/(\varkappa-1)}.$$

For the value of $\varkappa$ = 1.4, we have

$$F = \rho^{-7/5}\left(1 + \frac{1}{3}\rho^3 - \rho^5\right) + \frac{1}{2}\ln\frac{1-\rho^{1/5}}{1+\rho^{1/5}},$$
$$f = \rho^{1/5}\left(1 + \frac{1}{3}\rho^3 + \frac{1}{5}\rho^5\right) + \frac{1}{2}\ln\frac{1-\rho^{1/5}}{1+\rho^{1/5}},$$
$$F' = (1 - 6V^2)/V\rho^{7/5}, \qquad f' = \rho/V, \qquad \rho = (1 - V^2)^{5/2}.$$

The region of flow will be represented as a zone, bounded by stream lines ψ = 0 (axis) and $\psi = \psi_N$ (nozzle wall), in which we shall assume that ψ_N = 1. Investigating the arbitrary N-th approximation in the method of integral relations, let us divide this region into N equal strips with the help of the system of stream lines $\psi = \psi_n$ = n/N, n = 0, 1, ..., N. We will designate the values of all the functions on the stream line $\psi = \psi_n$ by the index n; then the index N will correspond to the wall of the nozzle. Let us integrate equations (1) with respect to ψ from ψ_0 = 0 to each stream line $\psi = \psi_n$, n = 1, 2, ..., N. As a result, we will arrive at a system of 2N independent integral relations

$$\frac{d}{d\varphi}\int_0^{\psi_n} F\, d\psi + \vartheta_n = 0, \qquad \frac{d}{d\varphi}\int_0^{\psi_n} \vartheta\, d\psi - f_n + f_0 = 0 \qquad (n = 1, 2, \ldots, N). \tag{2}$$

The functions F and ϑ, appearing under the integral sign, will be determined by approximation through interpolation by polynomials along ψ from base points of interpolation on the stream lines $\psi = \psi_n$. With the help of symmetry for these functions, let us write

$$F = \sum_{n=0}^{N} \omega_{2n}(\varphi)\, \psi^{2n}, \qquad \vartheta = \sum_{n=1}^{N} \chi_{2n-1}(\varphi)\, \psi^{2n-1}, \tag{3}$$

where

$$\omega_{2n} = \sum_{k=0}^{N} a_{2n,k} F_k, \qquad \chi_{2n-1} = \sum_{k=1}^{N} b_{2n-1,k} \vartheta_k,$$

in which $a_{2n,k}$ and $b_{2n-1,k}$ are the index coefficients. Then the integral relations (2) will be reduced to an approximation system of 2N ordinary differential equations in φ.

The equation of the nozzle contour will be given in the form $\vartheta = \vartheta_N(s)$, where s is the dimensionless length of the arc measured from the critical section along the contour of the nozzle in the direction of flow. From the independent variable φ in the approximational system, it is more convenient to change to the variable s, keeping in mind the relation $d\varphi/ds = V_N$.

The approximational system of ordinary differential equations is divisible into two subsystems, which are found from the first and second groups of integral relations and comprise, correspondingly, the derivatives dV_n/ds, $n = 0, 1, \ldots\ldots, N$, and $d\vartheta_n/ds$, $n = 1, 2, \ldots, N$. / Insofar as ω_{2n} and χ_{2n-1} in expressions (3) stand for linear combinations, correspondingly, the values of F_n and ϑ_n on the streamlines under consideration, as well as the derivatives dV_n/ds and $d\vartheta_n/ds$, will enter linearly in these subsystems, and the aforementioned can be written as follows:

$$\sum_{k=0}^{N} A_{nk} F_k \frac{dV_k}{ds} = -\vartheta_n V_N \quad (n = 1, 2, \ldots, N), \tag{4}$$

$$\sum_{k=1}^{N} B_{nk} \frac{d\vartheta_k}{ds} = (f_n - f_0) V_N \quad (n = 1, 2, \ldots, N), \tag{5}$$

where A_{nk}, B_{nk} are the numerical values of the generalized coefficients of Kotes for the integrals, calculable in relation to the interpolational polynomials (3). /1132

Subsystem (5) is resolved in relation to the derivatives and gives

$$\frac{d\vartheta_n}{ds} = V_N \sum_{k=0}^{N} C_{nk} f_n \quad (n = 1, 2, \ldots, N-1), \tag{6}$$

$$\frac{d\vartheta_N}{ds} = V_N \sum_{k=0}^{N} C_{Nk} f_k, \tag{7}$$

where the C_{nk} are the numerical coefficients.

Since the $\vartheta_N(s)$ are given, the identity (7) becomes the final relationship. Differentiating (7) along s, we will get, closing the subsystem (4), the equation:

$$\sum_{k=0}^{N-1} C_{Nk} f'_k \frac{dV_k}{ds} + \left(C_{NN} f'_N + \frac{1}{V_N^2} \frac{d\vartheta_N}{ds} \right) \frac{dV_N}{ds} = \frac{1}{V_N} \frac{d^2\vartheta_N}{ds^2}. \tag{8}$$

Now solving (4) and (8), we will get

$$\frac{dV_n}{ds} = \frac{\Delta_n}{\Delta} \qquad (n = 0, 1, \ldots, N). \tag{9}$$

In this manner, differential equations (6) and (9) represent the solution of the approximating system for determination of the velocities V_n and the angles of inclination ϑ_n on the streamlines $\psi = \psi_n$. This system is numerically integrated on an electronic computer.

The determinants Δ and Δ_n, appearing in (9), can be written without difficulty in the general form for the N-th approximation and are expressed through F'_n, f'_n, V_N, ϑ_N, $d\vartheta_N/ds$ and $d^2\vartheta_N/ds^2$. Let us find the values of the coefficients $a_{2n,k}$, $b_{2n-1,k}$, A_{nk}, C_{nk} for the first and second approximations. For $N = 1$ we have $a_{01} = 0$, $a_{00} = -a_{20} = a_{21} = b_{11} = 1$, $A_{10} = 2/3$, $A_{11} = 1/3$, $C_{10} = -C_{11} = -2$. For $N = 2$, the corresponding coefficients are given in the table.

k	$3a_{2n,k}$			$3b_{2n-1,k}$		$180A_{nk}$		$3C_{nk}$	
	$n=0$	1	2	1	2	1	2	1	2
0	3	−15	12	—	—	57	24	−9	18
1	0	16	−16	8	−8	34	128	8	−32
2	0	−1	4	−1	4	−1	28	1	14

Investigations show that in the N-th approximation, the determinant Δ becomes zero N times in the transonic region of the nozzle under the conditions of attainment of sonic velocities on the streamlines N of singular moving saddle points. The appearance of such singular points (9) is determined by the fact that in the scheme of solution under discussion there appears a supersonic (hyperbolic) region, where it is impossible to fix normal boundary conditions. The requirements of regularity of solution in these singular points is comprised in the fact that, in each of them, for example, there must be

$$\Delta_N = 0 \text{ when } \Delta = 0 \tag{10}$$

by virtue of which then, automatically, all the other $\Delta_n = 0$. The indefinitiveness of derivatives dV_n/ds at the singular points is determined in (9) by the usual method.

The infinitely distant point ($s \to -\infty$) also appears as a singular point of the approximating system. Here it is necessary to satisfy specific conditions, for example, for a nozzle with parallel walls when $s \to -\infty$ it is necessary that all the $\vartheta_n \to 0$ and all the $V_n \to V_\infty$, where V_∞ is the sought for value of velocity of a uniform flow at the entrance of the nozzle.

In this manner, for the approximational system of equations, (6) and (9) have place for a boundary problem which can be solved by a choice of either of two different methods:

1) integrating this system in the direction from $s = -\infty$ toward singular points in the transonic region;

2) integrating the system from one of these singular points to $s = -\infty$ /1133 and to the other singular points.

It turns out that at infinity the approximational system has N arbitrary parameters (in the approximation N = 1, one arbitrary parameter -- the magnitude of gas expansion through the nozzle, i.e., for a nozzle with parallel walls at infinity, simply the value of V_∞). In the first case, when the integration of the system proceeds from infinity, these N parameters are selected in such a way as to satisfy the conditions (10) at N singular points. This means that here it is necessary to solve a boundary problem of N-th order.

In the second case, numerical integration starts at any singular point, where three end conditions (7) and (10) allow us to express any three quantities, for example, the coordinate s (i.e., ϑ_N) and two velocities. The remaining 2N-2 sought for quantities are picked up through N-1 conditions of regularity at the rest of the singular points and through N-1 conditions at infinity. Consequently, in this case, it becomes necessary to solve a boundary problem with 2N-2 conditions. When N = 1 and N = 2, this route for solution turns out to be more convenient that the first method; when N = 1, we simply come up with the problem of Cauchy and all the flows are contained in the result of only the singlefold integrable system of equations (6) and (9).

In the practical resolution of the calculations, instead of $s = -\infty$ we investigate any sufficiently large negative value of s_*, during which we utilize an asymptotic solution of the approximating system. The satisfaction of the conditions at infinity can be replaced by the requirement that the solution resulting from the method of integral relations satisfies, under this value of s_*, the linearized equation of Prandtl (as was done in [3]).

Let us also note that after the determination of the result, we can, using formulas (3), construct a velocity field and, specifically, the sonic line and characteristic bounding the region of influence. The calculation of flow in the supersonic region of the nozzle is expediently carried out by the method of characteristics [4] under the given conditions, obtained by the method of integral relations on certain supersonic lines, φ = const, lying immediately beyond the region of influence.

As an illustration of the described method, the flow in the subsonic and transonic parts of a nozzle were worked out; the given equation is

$$\vartheta_N = a\,\mathrm{sh}\,bs\,/\,\mathrm{ch}^3\,bs.$$

For values representing the parameters a = 0.75 and b = 0.125, the calculations were carried out for the first and second approximations. The results of the calculation are shown in Figure 1, which shows the change in the velocity /1134

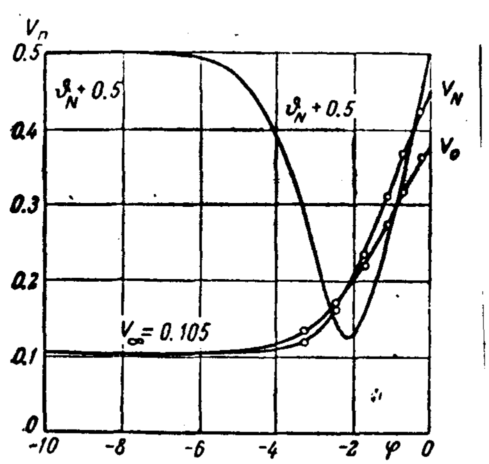

Figure 1.

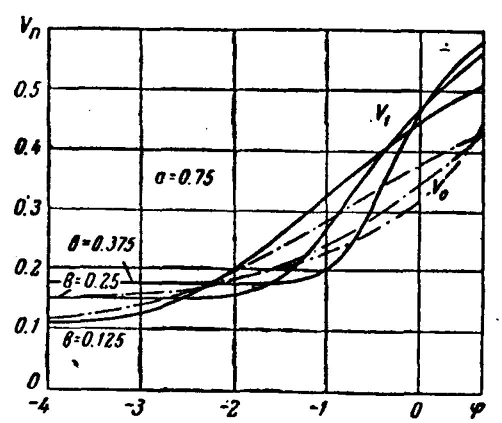

Figure 2.

along the axis, V_0, and along the wall, V_N, of the nozzle depending on the potential φ. Here the continuous curves correspond to N = 1, and the dotted curves to N = 2; the value φ = 0 corresponds to the critical section of the nozzle. The very same illustration shows a curve relating ϑ_N and φ, which characterize the geometry of the nozzle. As can be seen, in this case even the approximation N = 1 gives good accuracy.

With the first approximation, the flow was calculated for a series of nozzles with various geometries (a = 0.75, b = 0.125, 0.25, 0.375). For these nozzles, Figure 2 gives the derived velocities V_0 (dotted and stroked line) and

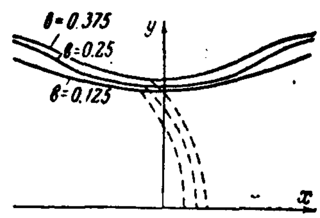

Figure 3.

V_1 (solid line), while Figure 3 gives the form of the sonic line (dotted).

In conclusion, let us note that the given method may be extended to the case of an axisymmetric nozzle, where an analogous scheme of solution is adopted, but the functions F, ϑ, and also y are approximated in steps of $\sqrt{\psi}$. Besides this, the system (4) — (5) allows independent construction of the wall of the nozzle for a given distribution of velocities along its axis; nevertheless, it is more convenient here to use a different method of solution.

To editing on June 22, 1963.

REFERENCES

1. Dorodnitsyn, A. A.: Ob odnom metode chislennogo resheniya nekotorykh nelineynykh zadach aerogidrodinamiki. (Concerning one method of numerical solution of certain nonlinear aerohydrodynamic problems.) Tr. III, Vses. matem. s"ezda, Vol. 3, 1956, Moscow; Izd-vo AN SSSR, pp. 447-453, 1958.

2. Belotserkovskiy, O. M. and P. I. Chushkin: Chislennyy metod integral'nykh sootnosheniy. (Numerical method of integral relations.) Zh. vychisl. matem. i matem. fiz. 2(5): 731-759, 1962.

3. Chushkin, P. I.: Obtekaniye ellipsov i ellipsoidov dozvukovym potokom gaza. (Flow around ellipses and ellipsoids of a subsonic stream of gas.) V. sb. "Vychisl. matem.", Isd-vo AN SSSR 2: 20-44, 1956.

4. Katskova, O. N., I. N. Naumova, Yu. D. Shmyglevskiy and N. P. Shulishnina: Opyt rascheta ploskikh i osesimmetrichnykh sverkhzvukovykh techeniy gaza metodom kharakteristik. (An experiment in the calculation of plane and axisymmetric supersonic flows of gas by the method of characteristics.) VTs AN SSSR, Moscow, 1961.

FRANK C. FARNHAM COMPANY
133 South 36th Street
Philadelphia, Pa. 19104